Nonlinear Unknown Input Observability: Analytical expression of the observable codistribution in the case of a single unknown input

Agostino Martinelli

Nonlinear Unknown Input Observability: Analytical expression of the observable codistribution in the case of a single unknown input

Agostino Martinelli*†

Abstract

This paper investigates the unknown input observability problem in the nonlinear case. Specifically, the systems here analyzed are characterized by dynamics that are nonlinear in the state and linear in the inputs and characterized by a single unknown input and multiple known inputs. Additionally, it is assumed that the unknown input is a differentiable function of time (up to a given order). The goal of the paper is not to design new observers but to provide a simple analytic condition in order to check the weak local observability of the state. In other words, the goal is to extend the well known observability rank condition to these systems. Specifically, the paper provides a simple algorithm to directly obtain the entire observable codistribution. As in the standard case of only known inputs, the observable codistribution is obtained by recursively computing the Lie derivatives along the vector fields that characterize the dynamics. However, in correspondence of the unknown input, the corresponding vector field must be suitably rescaled. Additionally, the Lie derivatives must be computed also along a new set of vector fields that are obtained by recursively performing suitable Lie bracketing of the vector fields that define the dynamics. In practice, the entire observable codistribution is obtained by a very simple recursive algorithm. However, the analytic derivations required to prove that this codistribution fully characterizes the weak local observability of the state are complex and, for the sake of brevity, are provided in a separate technical report. The proposed analytic approach is illustrated by checking the weak local observability of several nonlinear systems driven by known and unknown inputs.

1 INTRODUCTION

The problem of state observability for systems driven by unknown inputs (UI) is a fundamental problem in control theory. This problem was introduced and firstly investigated in the seventies [3, 4, 10, 22]. A huge effort has then been devoted to design observers for both linear and nonlinear systems in presence of UI, e.g., [1, 2, 5, 6, 7, 8, 9, 11, 12, 14, 16, 17, 23].

The goal of this paper is not to design new observers for systems driven by UI but to provide a simple analytic condition in order to check the weak local observability of the state. The obtained results hold for systems whose dynamics are nonlinear in the state and linear in both the known and the unknown inputs. Additionally, we restrict the analysis to the case of a single unknown input (or disturbance) and we assume that this unknown input is a differentiable function of time (up to a given order).

In [13] the observability properties of a nonlinear system are derived starting from the definition of indistinguishable states. In section 2 we introduce a new definition of *indistinguishable states* for the case UI. Then, in section 3 we introduce the approach to check if the state is weakly locally observable at a given point. For the brevity sake, we only provide the method. All the analytical derivations are provided separately in a technical report [21].

As in the standard case of only known inputs, the observable codistribution is obtained by recursively computing the Lie derivatives along the vector fields that characterize the dynamics. However, in correspondence of the unknown input (denoted with w), the corresponding vector field (denoted with g) must be suitably rescaled. In particular, it must be divided by the first order Lie derivative of the output along g (the result is the vector field $\frac{g}{L^1_g}$ that appears in the second line of the algorithm in definition 2). Additionally, the Lie derivatives must be computed also along a new set of vector fields that are obtained by recursively performing suitable Lie bracketing of the vector fields that define the dynamics (the vectors ϕ^i_m in definition 2). In practice, the entire observable codistribution is obtained by a very simple recursive algorithm. However, the analytic derivations required to prove that this codistribution fully characterizes the weak local observability of the state are complex and, for the sake of brevity, are provided in a separate technical report [21]. Finally, the recursive algorithm converges in a finite number of steps and the criterion to establish that the convergence has been reached is provided.

*This work was supported by the French National Research Agency ANR through the project VIMAD

†A. Martinelli is with INRIA Rhone Alpes, Montbonnot, France e-mail: agostino.martinelli@ieee.org

The proposed analytic approach is illustrated in section 4 by checking the weak local observability of several nonlinear systems driven by a single unknown input.

2 The considered system

In the sequel we will refer to a nonlinear control system with m_u known inputs ($u \equiv [u_1, \cdots, u_{m_u}]^T$) and a single unknown input or disturbance (w). The state is the vector $x \in M$, with M an open set of $\mathbb{R}^n$. We assume that the dynamics are nonlinear with respect to the state and linear with respect to the inputs (both known and unknown). Finally, for the sake of simplicity, we will refer to the case of a single output y (the extension to multiple outputs is straightforward). Our system is characterized by the following equations:

$$\begin{cases} \dot{x} = \sum_{i=1}^{m_u} f_i(x) u_i + g(x) w \\ y = h(x) \end{cases} \tag{2.1}$$

where $f_i(x)$, $i = 1, \cdots, m_u$, and $g(x)$ are vector fields in M and the function $h(x)$ is a scalar function defined on the open set M. For the sake of simplicity, we will assume that all these functions are analytic functions in M.

Let us consider the time interval $\mathcal{I} \equiv [0,\ T]$. Note that, since the equations in (2.1) do not depend explicitly on time, this can be considered as a general time interval of length T. In the sequel, we will assume that the solution of (2.1) exists in $\mathcal{I}$ and we will denote by $x(t;\ x_0;\ u;\ w)$ the state at a given time $t \in \mathcal{I}$, when $x(0) = x_0$ and the known input and the disturbance are $u(t)$ and $w(t)$, respectively, $\forall t \in \mathcal{I}$.
We introduce the following definition:

Definition 1 (Indistinguishable states) *Two states x_a and x_b are indistinguishable if, for any $u(t)$ (the known input vector function), there exist $w_a(t)$ and $w_b(t)$ (i.e., two unknown inputs in general, but not necessarily, different from each other) such that $h(x(t;\ x_a;\ u;\ w_a)) = h(x(t;\ x_b;\ u;\ w_b))\ \forall t \in \mathcal{I}$.*

This definition states that, if x_a and x_b are indistinguishable, then, for any known input, by looking at the output during the time interval $\mathcal{I}$, we cannot conclude if the initial state was x_a and the disturbance w_a or if the initial state was x_b and the disturbance w_b. We remark that, contrary to the definition of indistinguishable states in the case without disturbances, the new definition does not establish an equivalence relation. Indeed, we can have x_a and x_b indistinguishable, x_b and x_c indistinguishable but x_a and x_c are not indistinguishable. As in the case of known inputs, given x_0, the indistinguishable set I_{x_0} is the set of all the states x such that x and x_0 are indistinguishable. Starting from this definition, we can use exactly the same definitions of observability and weak local observability adopted in the case without disturbances.

3 The observable codistribution Ω

This section introduces the analytical method to check the weak local observability of the state x that satisfies (2.1) at a given $x_0 \in M$. This is obtained by computing a codistribution (the observable codistribution). In the sequel, we will denote by L_g^1 the first order Lie derivative of the function $h(x)$ along the vector field $g(x)$, i.e.,

$$L_g^1 \equiv \mathcal{L}_g h$$

The analytical computation of the observable codistribution is based on the assumption that $L_g^1 \neq 0$ on a given neighbourhood of x_0. In the appendix we show that, when this assumption does not hold, it is possible to introduce new local coordinates and the observability properties can be investigated starting from a new output that satisfies the aforementioned assumption.
We will denote with the symbol d_x the gradient with respect to the state x. Additionally, for a given codistribution Λ and a given vector field θ, we will denote by $\mathcal{L}_\theta \Lambda$ the codistribution whose covectors are the Lie derivatives along θ of the covectors in Λ (we are obviously assuming that the dimension of these covectors coincides with the dimension of θ). Finally, given two vector spaces V_1 and V_2, we denote with $V_1 + V_2$ their sum, i.e., the span of all the generators of both V_1 and V_2. We define the Ω codistribution recursively, as follows

Definition 2 (Ω codistribution) *This codistribution is defined recursively by the following algorithm:*

1. $\Omega_0 = d_x h$;
2. $\Omega_m = \Omega_{m-1} + \sum_{i=1}^{m_u} \mathcal{L}_{f_i} \Omega_{m-1} + \mathcal{L}_{\frac{g}{L_g^1}} \Omega_{m-1} + \sum_{i=1}^{m_u} \mathcal{L}_{\phi_{m-1}^i} d_x h$

where the vectors $\phi_m^i \in \mathbb{R}^n$ ($i = 1, \cdots, m_u$) are defined by the following algorithm:

1. $\phi_0^i = f_i$;
2. $\phi_m^i = \frac{[\phi_{m-1}^i,\ g]}{L_g^1}$

where the parenthesis $[\cdot, \cdot]$ denote the Lie brackets of vector fields.

In [21] we prove that the first algorithm in definition 2 converges. The proof is carried out by first considering the case of a single known input (i.e., $m_u = 1$) and is given by theorem 2 in section 7. Then, its validity is extended to the case of multiple inputs ($m_u > 1$) in section 8. In [21] it is also provided the criterion to check that the convergence has been reached. This criterion needs first of all to compute the following function:

$$\rho = \frac{\mathcal{L}_g^2 h}{(L_g^1)^2}$$

In [21] we prove that it exists m' such that $d_x\rho \in \Omega_{m'}$ (and therefore $d_x\rho \in \Omega_m \ \forall m \geq m'$). Additionally, we prove that the convergence of the algorithm has been reached when $\Omega_{m+1} = \Omega_m$, and $m \geq m'$. From the derivations in [21] it is possible to see that the required number of steps is at most $2n + 2$.

In [21] it is also shown that the computed codistribution is the entire observable codistribution. Also in this case, the proof is given by first considering the case of a single known input (see theorem 1 in section 6) and then, its validity is extended to the case of multiple inputs in section 8. Note that this proof is based on the assumption that the unknown input (w) is a differentiable function of time, up to a given order (the order depends on the specific case).

We conclude this section by outlining the steps to investigate the weak local observability at a given point x_0 of a nonlinear system driven by a single disturbance and several known inputs. In other words, to investigate the weak local observability of a system defined by a state that satisfies the dynamics in (2.1). The validity of the following procedure is a consequence of the theoretical results obtained in [21]:

1. For the chosen x_0, compute $L_g^1 (= \mathcal{L}_g^1 h)$ and $\rho \left(= \frac{\mathcal{L}_g^2 h}{(L_g^1)^2}\right)$. In the case when $L_g^1 = 0$, introduce new local coordinates, as explained in the appendix and re-define the output.

2. Build the codistribution Ω_m (at x_0) by using the algorithm provided in definition 2, starting from $m = 0$ and, for each m, check if $d_x\rho \in \Omega_m$.

3. Denote by m' the smallest m such that $d_x\rho \in \Omega_m$.

4. For each $m \geq m'$ check if $\Omega_{m+1} = \Omega_m$ and denote by $\Omega^* = \Omega_{m^*}$ where m^* is the smallest integer such that $m^* \geq m'$ and $\Omega_{m^*+1} = \Omega_{m^*}$ (note that $m^* \leq 2n + 2$).

5. If the gradient of a given state component (x_j, $j = 1, \cdots, n$) belongs to Ω^* (namely if $d_x x_j \in \Omega^*$) on a given neighbourhood of x_0, then x_j is weakly locally observable at x_0. If this holds for all the state components, the state x is weakly locally observable at x_0. Finally, if the dimension of Ω^* is smaller than n on a given neighbourhood of x_0, then the state is not weakly locally observable at x_0.

4 Applications

We apply the method described in section 3 in order to investigate the observability properties of several nonlinear systems characterized by the equations given in (2.1).

We consider a vehicle that moves on a $2D$-environment. The configuration of the vehicle in a global reference frame, can be characterized through the vector $[x_v, \ y_v, \ \theta]^T$ where x_v and y_v are the cartesian vehicle coordinates, and θ is the vehicle orientation. We assume that the dynamics of this vector satisfy the unicycle differential equations:

$$\left[\begin{aligned} \dot{x}_v &= v\cos\theta \\ \dot{y}_v &= v\sin\theta \\ \dot{\theta} &= \omega \end{aligned}\right. \tag{4.2}$$

where v and ω are the linear and the rotational vehicle speed, respectively, and they are the system inputs. We consider the following three cases of output (see also figure 1 for an illustration):

1. the distance from the origin (e.g., a landmark is at the origin and its distance is measured by a range sensor);

2. the bearing of the origin in the local frame (e.g., a landmark is at the origin and its bearing angle is measured by an on-board camera);

3. the bearing of the vehicle in the global frame (e.g., a camera is placed at the origin).

We can analytically express the output in terms of the state. We remark that the expressions become very simple if we adopt polar coordinates: $r \equiv \sqrt{x_v^2 + y_v^2}$, $\phi = atan\frac{y_v}{x_v}$. We have, for the three cases, $y = r$, $y = \beta = \pi - (\theta - \phi)$ and $y = \phi$, respectively. For each of these three cases, we consider the following two cases: v is known, ω is unknown; v is unknown, ω is known. The dynamics in these new coordinates become:

$$\left[\begin{aligned} \dot{r} &= v\cos(\theta - \phi) \\ \dot{\phi} &= \frac{v}{r}\sin(\theta - \phi) \\ \dot{\theta} &= \omega \end{aligned}\right. \tag{4.3}$$

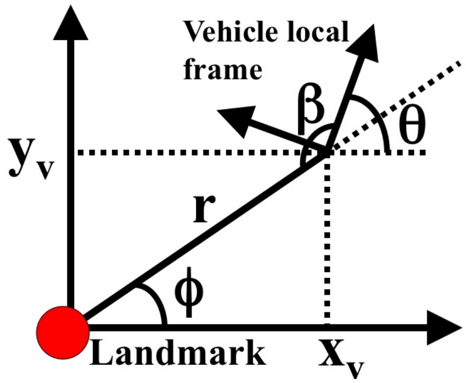

Figure 1: The vehicle state in cartesian and polar coordinates together with the three considered outputs.

4.1 $y = r$, $u = \omega$, $w = v$

In this case we have

$$f = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad g = \begin{bmatrix} \cos(\theta-\phi) \\ \frac{\sin(\theta-\phi)}{r} \\ 0 \end{bmatrix}$$

We follow the five steps mentioned at the end of section 3. We have $L^1_g = \cos(\theta - \phi)$ and $\rho \equiv \frac{L^2_g}{(L^1_g)^2} = \frac{\tan^2(\theta-\phi)}{r}$. Additionally:

$$d_x\rho = \frac{\tan(\theta-\phi)}{r}\left[-\frac{\tan(\theta-\phi)}{r}, \; -\frac{2}{\cos^2(\theta-\phi)}, \; \frac{2}{\cos^2(\theta-\phi)}\right]$$

We also have $\Omega_0 = span\{[1, 0, 0]\}$. Hence, $d_x\rho \notin \Omega_0$. Additionally, $\Omega_1 = \Omega_0$. We need to compute Ω_2 and, in order to do this, we need to compute ϕ_1. We obtain: $\phi_1 = \begin{bmatrix} -\tan(\theta-\phi) \\ \frac{1}{r} \\ 0 \end{bmatrix}$ and $\Omega_2 = span\left\{[1, 0, 0], \; \left[0, \frac{1}{\cos^2(\theta-\phi)}, -\frac{1}{\cos^2(\theta-\phi)}\right]\right\}$. It is immediate to check that $d_x\rho \in \Omega_2$, meaning that $m' = 2$. Additionally, by a direct computation, it is possible to check that $\Omega_3 = \Omega_2$ meaning that $m^* = 2$ and $\Omega^* = \Omega_2$, whose dimension is 2. We conclude that the dimension of the observable codistribution is equal to 2 and the state is not weakly locally observable.

4.2 $y = r$, $u = v$, $w = \omega$

In this case we have

$$f = \begin{bmatrix} \cos(\theta-\phi) \\ \frac{\sin(\theta-\phi)}{r} \\ 0 \end{bmatrix} \quad g = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

We follow the five steps mentioned at the end of section 3. We easily obtain $L^1_g = 0$. Hence, we have to introduce new local coordinates, as explained in the appendix. We obtain $\mathcal{L}^1_f h = \cos(\theta - \phi)$ and we obtain that the relative degree of the associated system in (5.6) is $r = 2$. Let us denote the new coordinates by x'_1, x'_2, x'_3. In accordance with (5.7) and (5.8) we should set $x'_1 = r$ and $x'_2 = \cos(\theta - \phi)$. On the other hand, to simplify the computation, we set $x'_2 = \theta - \phi$. Finally, we set $x'_3 = \theta$. We compute the new vector fields that characterize the dynamics in the new coordinates. We have:

$$\tilde{f} \equiv \begin{bmatrix} \cos(x'_2) \\ -\frac{\sin(x'_2)}{x'_1} \\ 0 \end{bmatrix} \qquad \tilde{g} \equiv \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \tag{4.4}$$

Additionally, we set $\tilde{h} = \cos(x'_2)$ and $\Omega_1 = span\{[1, 0, 0], \; [0, -\sin(x'_2), 0]\}$. In the new coordinates we obtain: $L^1_g = -\sin(x'_2)$ and $\rho = -\frac{\cos(x'_2)}{\sin^2(x'_2)}$. It is immediate to check that $d_x\rho \in \Omega_1$, meaning that $m' = 1$. Additionally, by a direct computation, it is possible to check that $\Omega_2 = \Omega_1$ meaning that $m^* = 1$ and $\Omega^* = \Omega_1$, whose dimension is 2. We conclude that the dimension of the observable codistribution is equal to 2 and the state is not weakly locally observable.

4.3 $y = \theta - \phi$, $u = \omega$, $w = v$

In this case we have

$$f = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad g = \begin{bmatrix} \cos(\theta-\phi) \\ \frac{\sin(\theta-\phi)}{r} \\ 0 \end{bmatrix}$$

We follow the five steps mentioned at the end of section 3. We have $L^1_g = -\frac{\sin(\theta-\phi)}{r}$ and $\rho = 2\cot(\theta - \phi)$. Additionally:

$$d_x\rho = \frac{2}{\sin^2(\theta-\phi)}[0, 1, -1]$$

We also have $\Omega_0 = span\{[0, -1, 1]\}$. Hence, $d_x\rho \in \Omega_0$, meaning that $m' = 0$. Additionally, by a direct computation, it is possible to check that $\Omega_1 = \Omega_0$ meaning that $m^* = 0$ and $\Omega^* = \Omega_0$, whose dimension is 1. We conclude that the dimension of the observable codistribution is equal to 1 and the state is not weakly locally observable.

4.4 $y = \theta - \phi$, $u = v$, $w = \omega$

In this case we have

$$f = \begin{bmatrix} \cos(\theta-\phi) \\ \frac{\sin(\theta-\phi)}{r} \\ 0 \end{bmatrix} \quad g = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

We follow the five steps mentioned at the end of section 3. We have $L^1_g = 1$ and $\rho = 0$. Hence,

$d_x\rho = [0,0,0]$ and we do not need to check if $d_x\rho \in \Omega_m$. In other words, we can set $m' = 0$. By a direct computation we obtain: $\Omega_0 = span\{[0,-1,1]\}$, $\Omega_1 = span\left\{[0,-1,1],\ \left[-\frac{\sin(\theta-\phi)}{r^2}, -\frac{\cos(\theta-\phi)}{r}, \frac{\cos(\theta-\phi)}{r}\right]\right\}$. Additionally, we obtain $\Omega_2 = \Omega_1$, meaning that $m^* = 1$ and $\Omega^* = \Omega_1$, whose dimension is 2. We conclude that the dimension of the observable codistribution is equal to 2 and the state is not weakly locally observable.

4.5 $y = \phi$, $u = \omega$, $w = v$ In this case we have

$$f = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad g = \begin{bmatrix} \cos(\theta-\phi) \\ \frac{\sin(\theta-\phi)}{r} \\ 0 \end{bmatrix}$$

We follow the five steps mentioned at the end of section 3. We have $L^1_g = \frac{\sin(\theta-\phi)}{r}$ and $\rho = -2\cot(\theta-\phi)$. Additionally:

$$d_x\rho = \frac{2}{\sin^2(\theta-\phi)}\,[0,-1,1]$$

We also have $\Omega_0 = span\{[0,1,0]\}$. Hence, $d_x\rho \notin \Omega_0$. Additionally, $\Omega_1 = \Omega_0$. We need to compute Ω_2 and, in order to do this, we need to compute ϕ_1. We obtain: $\phi_1 = \begin{bmatrix} -r \\ \cot(\theta-\phi) \\ 0 \end{bmatrix}$ and $\Omega_2 = span\left\{[0,1,0],\ \frac{1}{\sin^2(\theta-\phi)}\,[0,1,-1]\right\}$. It is immediate to check that $d_x\rho \in \Omega_2$, meaning that $m' = 2$. Additionally, by a direct computation, it is possible to check that $\Omega_3 = \Omega_2$ meaning that $m^* = 2$ and $\Omega^* = \Omega_2$, whose dimension is 2. We conclude that the dimension of the observable codistribution is equal to 2 and the state is not weakly locally observable.

4.6 $y = \phi$, $u = v$, $w = \omega$ In this case we have

$$f = \begin{bmatrix} \cos(\theta-\phi) \\ \frac{\sin(\theta-\phi)}{r} \\ 0 \end{bmatrix} \quad g = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

We follow the five steps mentioned at the end of section 3. We easily obtain $L^1_g = 0$. Hence, we have to introduce new local coordinates, as explained in the appendix. We obtain $\mathcal{L}^1_f h = \frac{\sin(\theta-\phi)}{r}$ and we obtain that the relative degree of the associated system in (5.6) is $r = 2$. Let us denote the new coordinates by $x'_1,\ x'_2,\ x'_3$. In accordance with (5.7) and (5.8) we set $x'_1 = \phi$ and $x'_2 = \frac{\sin(\theta-\phi)}{r}$. Finally, we set $x'_3 = \frac{\cos(\theta-\phi)}{r}$.

We compute the new vector fields that characterize the dynamics in the new coordinates. We obtain:

$$\tilde{f} \equiv \begin{bmatrix} x'_2 \\ -2x'_2x'_3 \\ x'^2_2 - x'^2_3 \end{bmatrix} \qquad \tilde{g} \equiv \begin{bmatrix} 0 \\ x'_3 \\ -x'_2 \end{bmatrix} \tag{4.5}$$

Additionally, we set $\tilde{h} = x'_2$ and $\Omega_1 = span\{[1,0,0],\ [0,1,0]\}$. In the new coordinates we obtain: $L^1_g = x'_3$ and $\rho = -\frac{x'_2}{x'^2_3}$. Since ρ depends on x'_3, $d_x\rho \notin \Omega_1$. Since the dimension of Ω_1 is already 2 and since we know that it exists a given integer m such that $d_x\rho \in \Omega_m$, we conclude that the dimension of Ω_m (and consequently the dimension of Ω^*) is larger than 2. Hence, the entire state is weakly locally observable.

We conclude this section by remarking that, the results obtained for the six cases previously investigated, agree with what we expected. By using the observability rank condition in [13], we easily obtain that, when both the inputs are known, the dimension of the observable codistribution is 2 for the first two observations ($y = r$ and $y = \theta - \phi$) and 3 for the last one ($y = \phi$). In particular, for the first two observations, all the initial states rotated around the vertical axis are indistinguishable. When one of the inputs misses, this unobservable degree of freedom obviously remains. Hence, for the first four cases previously investigated, it is non surprising that the dimension of the observable codistribution does not exceed 2. Additionally, for $y = \theta - \phi$ and $y = \phi$, when the first input (v) is unknown, we lose a further degree of freedom, which corresponds to the absolute scale.

5 CONCLUSIONS

In this paper we investigated the problem of nonlinear observability when one of the system inputs is unknown. The goal was not to design new observers but to provide simple analytic conditions in order to check the weak local observability of the state. An unknown input was also called disturbance. We introduced a simple analytic condition to check the weak local observability of the state at a given point x_0. This condition is based on the computation of a codistribution (the observable codistribution).

As in the standard case of only known inputs, the observable codistribution is obtained by recursively computing the Lie derivatives along the vector fields that characterize the dynamics. However, in correspondence of the unknown input (denoted with w), the corresponding vector field (denoted with g) must be suitably rescaled. In particular, it must be divided by the first order Lie derivative of the output along g at x_0 (the result is the vector field $\frac{g}{L^1_g}$ that appears in the second line of the algorithm in definition 2; note that when $L^1_g = 0$

at x_0, the coordinates must be changed, as explained in the appendix). Additionally, the Lie derivatives must be computed also along a new set of vector fields that are obtained by recursively performing suitable Lie bracketing of the vector fields that define the dynamics (the vectors ϕ^i_m in definition 2). In practice, the entire observable codistribution was obtained by a very simple recursive algorithm. However, the analytic derivations required to prove that this codistribution fully characterizes the weak local observability of the state are complex and, for the sake of brevity, they were provided in a separate technical report [21]. Finally, in [21] we prove that the recursive algorithm converges in a finite number of steps and the criterion to establish that the convergence has been reached was also provided. Also this proof is based on several tricky analytical steps (the details are provided in [21]). To this regard, we really wish to emphasize that almost all the properties derived in [21] have been proved by induction. Obtaining these properties, and not simply their proofs, has required a huge effort in terms of analytical computation and number of trials.

The proposed analytic approach has been illustrated by checking the weak local observability of several nonlinear systems driven by a single unknown input and a known input.

We are extending the analytic results presented in this paper to the case of multiple unknown inputs. So far, we have introduced a tool, *the extended observability rank condition*, that only provides sufficient conditions for the weak local observability in the case of multiple disturbances (see definition 2 and proposition 4 in [21]). In particular, we have successfully adopted this tool to investigate the observability properties of the visual-inertial structure from motion problem in the case of missing inputs [18, 19, 20].

APPENDIX

Let us suppose that $L^1_g = 0$ on a given neighbourhood of x_0. We introduce the following system associated with the system in (2.1):

$$\left\{\begin{array}{l} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{array}\right. \tag{5.6}$$

where $f(x) = f_{i'}(x)$ is one of the vector field $f_i(x)$ in (2.1). This is a system without disturbances and with a single known input u. Let us denote by r the relative degree of this system at x_0. Since $L^1_g = 0$ on a given neighbourhood of x_0, we have $r > 1$. Additionally, we can introduce the following new local coordinates (see proposition 4.1.3 in [15]):

$$x' = \mathcal{Q}(x) = \left[\begin{array}{c} \mathcal{Q}_1(x) \\ \cdots \\ \mathcal{Q}_n(x) \end{array}\right] \tag{5.7}$$

such that the first new r coordinates are:

$$\mathcal{Q}_1(x) = h(x),\ \mathcal{Q}_2(x) = \mathcal{L}^1_f h(x),\ \cdots,\ \mathcal{Q}_r(x) = \mathcal{L}^{r-1}_f h(x) \tag{5.8}$$

Now let us derive the equations of the original system (i.e., the one in (2.1)) in these new coordinates. We have:

$$\left\{\begin{array}{l} \dot{x}' = \sum_{i=1}^{m_u} \tilde{f}_i(x')u_i + \tilde{g}(x')w \\ y = x'_1 \end{array}\right. \tag{5.9}$$

where $\tilde{f}_i$ $(i = 1, \cdots, m)$ and $\tilde{g}$ can be analytically computed starting from the expression of $\mathcal{Q}_1, \cdots, \mathcal{Q}_n$. In particular, $\tilde{g}$ has the following structure:

$$\tilde{g} \equiv \left[\begin{array}{c} 0 \\ 0 \\ \cdots \\ 0 \\ \tilde{g}_r(x') \\ \cdots \\ \tilde{g}_n(x') \end{array}\right] \tag{5.10}$$

It is possible to check that the first r components of x' are weakly locally observable. Indeed, the first r components of x' are the output and its Lie derivatives along $\tilde{f}$ up to the $(r-1)-$order. Additionally, the first $(r-1)$ Lie derivatives are constant on the indistinguishable sets because all the Lie derivatives up to the $(r-1)-$order that includes at least one direction along $\tilde{g}$ vanish automatically (see propositions 1 and 3 in [21]). In order to investigate the observability properties of the remaining components, we consider the new output $\tilde{h}(x') = x'_r$ and we set $L^1_g = \tilde{g}_r = \mathcal{L}_g \mathcal{L}^{r-1}_f h \neq 0$.

References

[1] J.-P. Barbot, M. Fliess, T. Floquet, "An algebraic framework for the design of nonlinear observers with unknown inputs", IEEE Conf. on Decision and Control, New-Orleans, USA, 2007

[2] J.-P. Barbot, D. Boutat, T. Floquet ,"An observation algorithm for nonlinear systems with unknown inputs", Automatica, Volume 45, Issue 8, August 2009, Pages 1970-1974

[3] G. Basile and G. Marro. On the observability of linear, time invariant systems with unknown inputs. J. Optimization Theory Appl., 3:410–415, 1969.

[4] S.P. Bhattacharyya Observer design for linear systems with unknown inputs IEEE Transactions on Automatic Control, 23 (1978)

[5] Chen, W. and Saif, M. (2006). Unknown input observer design for a class of nonlinear systems: an LMI approach. In IEEE American Control Conference.

[6] Darouach, M., Zasadzinski, M., Xu, S. J. (1994). Full-order observers for linear systems with unknown inputs. IEEE Transactions on Automatic Control, 39(3)

[7] Floquet, T., Barbot, J. (2004). A sliding mode approach of unknown input observers for linear systems. In 43rd IEEE conference on decision and control. Atlantis, Paradise Island, Bahamas. December.

[8] Floquet, T., Edwards, C., Spurgeon, S. (2007). On sliding mode observers for systems with unknown inputs. Int. J. Adapt. Control Signal Process Vol 21, no. 89, 638–656, 2007.

[9] Guan, Y., Saif, M. (1991). A novel approach to the design of unknown input observers. IEEE Transactions on Automatic Control, 36(5).

[10] R. Guidorzi and G. Marro. On Wonham stabilizability condition in the synthesis of observers for unknown-input systems. Automatic Control, IEEE Transactions on, 16(5):499–500, oct 1971.

[11] Q.P. Ha, H. Trinh State and input simultaneous estimation for a class of nonlinear systems Automatica, 40 (2004).

[12] H. Hammouri and Z. Tmar. Unknown input observer for state affine systems: A necessary and sufficient condition. Automatica, 46(2):271–278, 2010.

[13] Hermann R. and Krener A.J., 1977, Nonlinear Controllability and Observability, Transaction On Automatic Control, AC-22(5): 728–740

[14] M. Hou, P.C. Mller Design of observers for linear systems with unknown inputs IEEE Transactions on Automatic Control, 37 (6) (1992).

[15] Isidori A., Nonlinear Control Systems, 3rd ed., Springer Verlag, 1995.

[16] Koenig, D. and Mammar, S. (2001). Design of a class of reduced order unknown inputs nonlinear observer for fault diagnosis. In IEEE American Control Conference.

[17] D. Koening, B. Marx, D. Jacquet Unknown input observers for switched nonlinear discrete time descriptor system IEEE Transactions on Automatic Control, 53 (1) (2008).

[18] A. Martinelli, Visual-inertial structure from motion: observability vs minimum number of sensors, International Conference on Robotics and Automation (ICRA 2014), May 2014, Hong Kong.

[19] A. Martinelli, Observability Properties and Deterministic Algorithms in Visual-Inertial Structure from Motion, Foundations and Trends in Robotics, 139–209, http://dx.doi.org/10.1561/2300000030, 2014

[20] A. Martinelli, Minimalistic sensor design in visual-inertial structure from motion, to be presented at the International Conference on Robotics and Automation, ICRA 2015, May 2015, Seattle.

[21] A. Martinelli, Non linear Unknown Input Observability: Extension of the observability rank condition and the case of a single unknown input, AUTOMATICA under review (available on line on the HAL archive https://hal.inria.fr/hal-01071314v4)

[22] S.H. Wang, E.J. Davison, P. Dorato Observing the states of systems with unmeasurable disturbance IEEE Transactions on Automatic Control, 20 (1975)

[23] F. Yang, R.W. Wilde Observer for linear systems with unknown inputs IEEE Transactions on Automatic Control, 33 (7) (1988).